

Chiral Alfvén Wave in Anomalous Hydrodynamics

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We study the hydrodynamic regime of chiral plasmas at high temperature. We find a new type of gapless collective excitation induced by chiral effects in an external magnetic field. This is a transverse wave, and it is present even in incompressible fluids, unlike the chiral magnetic and chiral vortical waves. The velocity is proportional to the coefficient of the gravitational anomaly. We briefly discuss the possible relevance of this “chiral Alfvén wave” in physical systems.

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Introduction.—Hot and/or dense matter with chirality imbalance is considered to be realized in a broad range of systems, including quark-gluon plasmas produced in heavy-ion collisions [1, 2], the early Universe [3, 4], compact stars and supernovae [5, 6], and Weyl semimetals [7–9]. The main reason why such chiral matter has attracted much attention recently is that it exhibits anomalous transport phenomena, related to quantum anomalies in field theory [10, 11]. Two prominent examples are the so-called chiral magnetic effect (CME) [2, 12–14] and chiral vortical effect (CVE) [15–19], which are the currents along the direction of a magnetic field and vorticity in chiral matter.

On the theoretical side, chiral (or anomalous) hydrodynamics [18] and chiral kinetic theory [20–23], which describe anomalous transport phenomena in chiral plasmas, have been formulated. The chiral plasmas have also been found to exhibit a new type of density wave in an external magnetic field or with a vorticity, called the chiral magnetic wave (CMW) [24, 25] and the chiral vortical wave (CVW) [26], as well as the unstable collective mode in the presence of dynamical (color) electromagnetic fields, called the chiral plasma instability (CPI) [27, 28] (see also Refs. [3, 4]). For recent numerical and analytical applications of chiral (magneto)hydrodynamics, see Ref. [29] and Refs. [30, 31], respectively.

In this paper, we show that there exists a new type of gapless collective excitation specific for charged chiral plasmas in an external magnetic field. Unlike the CMW and CVW, this is a transverse wave, and it is present even in incompressible fluids. This is somewhat similar to the Alfvén wave in normal charged plasmas, which propagates without compressing the medium, driven by magnetic tension forces [32]. As we will show, the velocity of this new mode is proportional to the coefficient of the mixed gauge-gravitational anomaly for chiral fermions.

We call it the “chiral Alfvén wave” (CAW). Since the CAW is the first example of the transverse wave with a vector-type perturbation induced by chiral effects, it should provide a unique signature in physical observables. We briefly discuss its possible phenomenological implications. Throughout the paper, we set $\hbar = c = e = 1$.

Physical argument.—Before going to a mathematical analysis based on chiral hydrodynamic equations, we first provide a physical argument as to why a new type of wave (which is different from the CMW and CVW) can exist for chiral fluids in an external magnetic field. We consider a fluid of single right-handed chiral fermions at high temperature T and zero chemical potential $\mu = 0$.

As shown in Fig. 1, we take the external magnetic field \mathbf{B} in the positive z direction and consider the perturbation of the local fluid velocity \mathbf{v} in the positive y direction, $\mathbf{v} = v(z)\hat{\mathbf{y}}$ with $\partial_z v(z) < 0$. For this local fluid velocity \mathbf{v} , the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ points in the positive x direction. In the chiral fluid, the vorticity induces the local chiral vortical current, $\mathbf{j} \propto T^2 \boldsymbol{\omega}$ [15, 19]. This current then receives the Lorentz force, $\mathbf{F} = \mathbf{j} \times \mathbf{B}$, in the negative y direction. Notice that this is the opposite direction as the original fluid velocity \mathbf{v} , so the Lorentz force acts

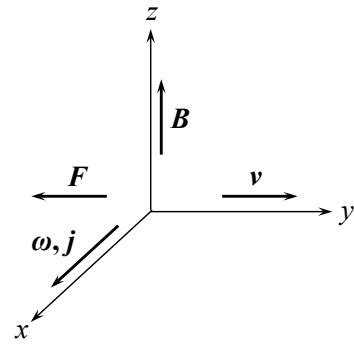


FIG. 1. Configuration of the vectors, \mathbf{B} , \mathbf{v} , $\boldsymbol{\omega}$, \mathbf{j} , and \mathbf{F} .

as a restoring force; it makes the perturbation of the fluid velocity \mathbf{v} oscillate, and this is the origin of a wave.

This argument holds even when the fluids are incompressible like the Alfvén wave in the normal charged fluids. This should be contrasted with sound waves in normal fluids and the CMW and CVW in chiral fluids, which can propagate only in compressible fluids. Apparently, the CVE specific for chiral fluids is essential for the presence of this wave—thus, the name chiral Alfvén wave.

In the following, we put this argument on a formal mathematical basis and derive its wave equation together with the explicit expression for the velocity of the CAW.

Chiral hydrodynamics.—Let us start with the generic hydrodynamics for plasmas of single right-handed chiral fermions in *external* electromagnetic fields. The hydrodynamic equations read [18]¹

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda, \quad (1)$$

$$\partial_\mu j^\mu = -CE^\mu B_\mu. \quad (2)$$

Here $T^{\mu\nu}$ is the energy-momentum tensor, j^μ is the electric current, $F^{\mu\nu}$ is the field strength, C is the anomaly coefficient [see Eq. (7) below], $E^\mu = F^{\mu\nu} u_\nu$, and $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$, with $u^\mu = \gamma(1, \mathbf{v})$ being the local fluid velocity.

In the Landau-Lifshitz frame, $T^{\mu\nu}$ and j^μ are given by [18]

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu} + \tau^{\mu\nu}, \quad (3)$$

$$j^\mu = nu^\mu + \xi\omega^\mu + \xi_B B^\mu + \nu^\mu, \quad (4)$$

with ϵ the energy density, P the pressure, n the charge density, and $\omega^\mu = \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta$ the vorticity.² The dissipative effects, such as the conductivity σ and viscosities, are incorporated in ν^μ and $\tau^{\mu\nu}$, which we ignore to simplify our argument for a moment. (We discuss these corrections later.) The transport coefficients ξ_B and ξ , corresponding to the CME [2, 12–14] and CVE [15–19], take the form [18, 33]

$$\xi_B = C\mu \left(1 - \frac{1}{2} \frac{n\mu}{\epsilon + P}\right) - \frac{D}{2} \frac{nT^2}{\epsilon + P}, \quad (5)$$

$$\xi = \frac{C}{2} \mu^2 \left(1 - \frac{2}{3} \frac{n\mu}{\epsilon + P}\right) + \frac{D}{2} T^2 \left(1 - \frac{2n\mu}{\epsilon + P}\right), \quad (6)$$

where T is the temperature and μ is the chemical potential. The transport coefficients C and D are related to the chiral anomaly and mixed gauge-gravitational anomaly, respectively, as [18, 19, 34–36]

$$C = \frac{1}{4\pi^2}, \quad D = \frac{1}{12}. \quad (7)$$

We now explicitly write down the hydrodynamic equations in an external magnetic field. We assume the bulk collective flow to be nonrelativistic, $|\mathbf{v}| \ll 1$, despite individual constituents of the fluid being relativistic. To be specific, we consider the following counting scheme: $\partial_t \sim O(\epsilon_t)$, $\nabla \sim O(\epsilon_s)$, and $\mathbf{v} \sim O(\delta)$ with three independent expansion parameters, $\epsilon_{s,t} \ll 1$ and $\delta \ll 1$. We take the gauge field A^μ to be the same order as T and μ , so that $\mathbf{B} \sim O(\epsilon_s)$.

As usual, it is convenient to consider the longitudinal and transverse projections of Eq. (1) with respect to the fluid velocity [37],

$$u_\nu \partial_\mu T^{\mu\nu} = u_\nu F^{\nu\lambda} j_\lambda, \quad (8)$$

$$(g_\nu^\rho - u^\rho u_\nu) \partial_\mu T^{\mu\nu} = (g_\nu^\rho - u^\rho u_\nu) F^{\nu\lambda} j_\lambda. \quad (9)$$

Although hydrodynamic equations to the linear order in \mathbf{v} will be sufficient for the purpose of deriving the wave equation of the CAW, we first write down a more generic “Euler equation” taking into account chiral effects. To this end, we keep the terms to the order of $O(\epsilon_t \delta, \epsilon_s \delta^2, \epsilon_s^2 \delta)$. Then, Eqs. (8), (9) for the temporal and spatial components ($\rho = 0, i$), and (2) are given by

$$(\partial_t + \mathbf{v} \cdot \nabla) \epsilon + (\epsilon + P) \nabla \cdot \mathbf{v} = 0, \quad (10)$$

$$(\mathbf{v} \cdot \nabla) P = 0, \quad (11)$$

$$(\epsilon + P)(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P - \mathbf{v}(\partial_t + \mathbf{v} \cdot \nabla) P + \mathbf{j} \times \mathbf{B}, \quad (12)$$

$$\partial_t n + \nabla \cdot \mathbf{j} = 0, \quad (13)$$

respectively. Here

$$\mathbf{j} = n\mathbf{v} + \xi\boldsymbol{\omega} + \xi_B \mathbf{B}, \quad (14)$$

with $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. The terms $\mathbf{v} \cdot (\mathbf{j} \times \mathbf{B})$ on the right-hand sides of Eqs. (10) and (11) are of order $O(\epsilon_s^2 \delta^2)$ and are ignored.

For plasmas with homogeneous and static ϵ , P , and n (which is the case where the variations of T and μ are much smaller and slower than \mathbf{v}), the hydrodynamic equations are further simplified to

$$(\epsilon + P)(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = (n\mathbf{v} + \xi\boldsymbol{\omega}) \times \mathbf{B}, \quad (15)$$

$$\nabla \cdot \mathbf{v} = 0. \quad (16)$$

¹ In this paper, we use the “mostly minus” metric signature $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

² Note that the definition of the vorticity ω^μ here is different from the one in Ref. [18] by a factor of 2.

Equation (16) is the incompressibility condition of fluids.

Chiral Alfvén wave.—We now assume homogeneous plasmas at high temperature $T \gg \mu$, where we can ignore the contribution of μ (and so n) [38]. Let us consider a small perturbation of \mathbf{v} , similarly to the analysis in sound waves in hydrodynamics and normal Alfvén waves in magnetohydrodynamics [32]. The linearized chiral hydrodynamic equation (15) in a magnetic field [to the order of $O(\epsilon_t \delta, \epsilon_s^2 \delta)$ by assuming $\delta \ll \epsilon_{s,t}$] is given by

$$(\epsilon + P)\partial_t \mathbf{v} = \xi \boldsymbol{\omega} \times \mathbf{B}, \quad (17)$$

where $\xi = DT^2/2$. Using $\boldsymbol{\omega} \times \mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{v} - \nabla(\mathbf{B} \cdot \mathbf{v})$, the above equation can be rewritten as

$$(\epsilon + P)\partial_t \mathbf{v} = \xi[(\mathbf{B} \cdot \nabla)\mathbf{v} - \nabla(\mathbf{B} \cdot \mathbf{v})]. \quad (18)$$

Taking the divergence of Eq. (18) and using Eq. (16), we have $\nabla^2(\mathbf{B} \cdot \mathbf{v}) = 0$. To satisfy this equation, we set

$$\mathbf{B} \cdot \mathbf{v} = 0, \quad (19)$$

i.e., the direction of \mathbf{v} is taken to be perpendicular to \mathbf{B} .

Without loss of generality, we take the magnetic field in the z direction, $\mathbf{B} = B\hat{z}$. From Eq. (18), we then obtain the wave equation

$$\partial_t \mathbf{v} = V_T \partial_z \mathbf{v}, \quad (20)$$

where

$$V_T = \frac{D}{2} \frac{T^2}{\epsilon + P} B. \quad (21)$$

Substituting the plane-wave solution of the form $\mathbf{v} = \mathbf{v}_0 e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}}$, we get the dispersion relation

$$\omega = -V_T k_z, \quad (22)$$

so the wave propagates in the opposite direction as the magnetic field. (For the fluid with left-handed chiral fermions, the wave propagates with the velocity V_T in the same direction as the magnetic field.) As the velocity of this collective mode is proportional to the coefficient D related to the mixed gauge-gravitational anomaly for chiral fermions, this phenomenon is specific for chiral fluids, and it is not present in normal charged fluids. This is the CAW. The incompressibility condition (16) means that the CAW is a transverse wave, $\mathbf{k} \cdot \mathbf{v} = 0$.

This clarifies the difference between the CAW and CMW, the latter of which is also a gapless collective excitation in chiral fluids with a magnetic field; the CMW is a longitudinal density wave and appears only in compressible fluids ($\partial_t n \neq 0$) [24, 25], while the CAW is

present even in incompressible fluids ($\partial_t n = 0$). Note also that the conventional Alfvén wave appears when the magnetic fields are dynamical [32]. In contrast, the CAW exists even when the magnetic fields are external.

For high-temperature chiral plasma, $\epsilon, P \propto T^4$, and the velocity of the CAW, given by Eq. (21), is $V_T \propto B/T^2$. On the other hand, the velocity of the CMW, given by $V_L \equiv B/(4\pi^2\chi)$ with $\chi \equiv \partial n/\partial \mu$ the susceptibility [25], is also $V_L \propto B/T^2$. Hence, V_T and V_L coincide up to (generally different) prefactors. Note that $V_{T,L} \ll 1$ in our counting scheme, where $\mathbf{B} \sim O(\epsilon_s)$.

When the dissipative effects are included, the terms $\nu \nabla^2 \mathbf{v}$ and $-\tilde{\sigma} B^2 \mathbf{v}$, with ν the kinematic viscosity and $\tilde{\sigma} = \sigma/(\epsilon + P)$, are also added to the right-hand side of Eq. (20). Then, the dispersion relation of Eq. (22) becomes

$$\omega = -V_T k_z - i(\nu k^2 + \tilde{\sigma} B^2) \quad (23)$$

with $k \equiv |\mathbf{k}|$. In this case, the CAW propagates with the velocity V_T , but is damped by dissipation.

Discussions.—Let us discuss phenomenological implications of the CAW. One possible physical situation is the chiral fluid in the early Universe. Above the electroweak phase transition temperature where the $SU(2)_L \times U(1)_Y$ symmetry is restored, the massless Abelian gauge field that can propagate at long distances is the hypermagnetic field associated with the $U(1)_Y$ hypercharge, rather than the ordinary magnetic field associated with the $U(1)_{EM}$ charge. As the hypermagnetic field couples differently to right- and left-handed electrons, the high-temperature plasma there is chiral [39].

One expects the emergence of the CAW in such chiral fluids with strong hypermagnetic fields, which should affect the polarization anisotropies of the cosmic microwave background radiation (CMBR). Since the CAW is the vector-type perturbation (i.e., the velocity $\delta \mathbf{v}$) while the CMW and CVW are scalar-type ones (i.e., the density δn), the former should leave a peculiar signature that can be distinguished from the latter, e.g., parity-odd correlations of multipole amplitudes with different angular momenta in the CMBR. We defer this question to future work.

It would be interesting to study the possible roles of the CAW in the turbulence of chiral magnetohydrodynamics relevant to the evolution of the primordial magnetic field of the Universe. One can also consider the analogue of the CAW in rotating chiral fluids (without external electromagnetic fields), which is relevant to the physics of

hot neutrino gas.

Finally, the CAW revealed in this paper may be understood in the language of chiral kinetic theory [20–23], in a way similar to Refs. [26, 40].

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